THE RAYLEIGH PROBLEM FOR A NON-NEWTONIAN ELECTRICALLY CONDUCTING FLUID

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The flow of an electrically conducting non-Newtonian fluid allowing for the effect of external electromagnetic fields was studied in [1-6, 9]. In particular, steady state Couette and Hartmann flow were considered in [4] for power-law non-Newtonian fluids and it was shown that dilatant and pseudoplastic fluids behave differently in an external magnetic field. Nonsteady-state self-similar flow of power-law non-Newtonian fluids without a magnetic field was considered in [7].

The present paper investigates the nonsteady-state self-similar flow of a power-law non-Newtonian electrically conducting fluid in an external magnetic field when a plate is set abruptly into uniform motion. The problem is solved for weak and for strong magnetic fields. It is shown that in dilatant fluids two regions arise separated by a moving boundary: the fluid at rest and the perturbed fluid. If uniformly exact approximations are obtained for pseudoplastic fluids by the method of expansion with respect to a small parameter, the solutions obtained by the same method for dilatant fluids have singularities in the neighborhood of the moving boundary. Thus, a solution is obtained in the latter case using the PLG (Poincare–Lighthill–Go) method and by the method of external and internal expansions [8]. The first two approximations to the solution are obtained for all the cases.

1. Let the half space $y > 0$ be filled with a power-law non-Newtonian electrically conducting incompressible fluid. At time $t = 0$, the plane $y = 0$ is abruptly set into a state of uniform motion with velocity U in the direction of the X axis. The magnetic Reynolds' number is small. The magnetic field strength vector H is parallel to the y axis.

In this case, the equations of motion of the fluid and the boundary conditions have the form

$$\frac{\partial u}{\partial t} - \frac{k}{\rho}\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right) = -\frac{\sigma\mu^2 H^2}{\rho}u$$
$$u(0,\ y) = 0 \quad u(t,\ 0) = U, \quad u(t,\ \infty) = 0, \quad t > 0 \tag{1.1}$$

Here u is the component of velocity along the x axis; ρ is the density; k and n are parameters in the law collecting surface tensions with deformation velocities for power-law fluids; σ is the electrical conductivity; and μ is the magnetic permeability.

For the self-similar problem, we require that

$$H = At^{-1/2} \quad (A = \text{const}) \tag{1.2}$$

We shall look for a solution of Eq. (1.1) in the form

$$u = Uf(\eta), \qquad \eta = y\,(nk\rho^{-1}tU^{n-1})^{-1/(n+1)} \tag{1.3}$$

We obtain the following equations and boundary conditions for determining the functions $f(\eta)$:

$$|f'|^{n-1}f'' + \frac{\eta}{n+1}f' = Nf, \quad f(0) = 1, \quad f(\infty) = 0$$
$$(N = \sigma\mu^2 A^2\rho^{-1} = \text{const}) \tag{1.4}$$

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A prime denotes the derivative of a function with respect to its argument. Equation (1.4) cannot be solved analytically for any value of N. Solutions are obtained below for small and large values of N using perturbation methods [8].

2. We look for a solution for the functions $f(\eta)$ in the case of small N in the form

$$f = f_0 + Nf_1 + N^2 f_2 + \ldots \tag{2.1}$$

For the zero-th approximation we have

$$f''_0 |f_0'|^{n-1} + \frac{\eta}{n+1} f_0' = 0, \quad f_0(0) = 1, \quad f_0(\infty) = 0 \tag{2.2}$$

The solution for f_0 has the form [7]

$$\begin{gathered} f_0 = 1 - \int_0^\eta \varphi^{\frac{1}{n-1}} d\xi, \qquad \varphi(\xi) = C_1 + \frac{1-n}{2(1+n)} \xi^2 \\ C_1 = \left\{ \left[\frac{1+n}{2(1-n)}\right]^{1/2} B\left(\frac{1}{2}, \frac{1+n}{2(1-n)}\right) \right\}^{2(1-n)/(1+n)} \quad \text{for} \quad n < 1 \\ C_1 = \left\{ \left[\frac{1+n}{2(n-1)}\right]^{1/2} B\left(\frac{1}{2}, \frac{n}{n-1}\right) \right\}^{2(1-n)/(1+n)} \quad \text{for} \quad n > 1 \end{gathered} \tag{2.3}$$

The function f_0 can be expressed in terms of the incomplete B-function [7] which can be written in terms of the hypergeometric function.

We note that in the case of a dilatant fluid (n > 1), for

$$0 \leqslant \eta \leqslant \left[\frac{2(n+1)}{n-1} C_1\right]^{1/2} = s_* \tag{2.4}$$

the fluid moves, while for $\eta \geq s_*$ it remains at rest, i.e., the perturbations penetrate gradually into the fluid. The effect of the magnetic field is not taken into account in the zero-th approximation.

The equation for determining f_1 has the form

$$\begin{gathered} |f_0'|^{n-1} f_1'' + \left[(1-n)|f_0'|^{n-2} f_0'' + \frac{\eta}{1+n}\right] f_1' = f_0 \\ f_1(0) = f_1(\infty) = 0 \end{gathered} \tag{2.5}$$

When the function $\varphi(\eta)$ is taken into account, the last equation can be rewritten in the form

$$\varphi f_1'' + \eta \frac{2-n}{1+n} f_1' = f_0 \tag{2.6}$$

Solving this equation with the appropriate boundary conditions for a pseudoplastic fluid (n < 1), we have

$$\begin{gathered} f_1 = \int_0^\eta \varphi^{\frac{2-n}{n-1}} \left(\int_0^\xi f_0 \varphi^{\frac{1}{1-n}} d\gamma - C_2 \right) d\xi \\ C_2 = \left(\int_0^\infty \varphi^{\frac{2-n}{n-1}} d\xi \right)^{-1} \int_0^\infty \left(\varphi^{\frac{2-n}{n-1}} \int_0^\xi f_0 \varphi^{\frac{1}{1-n}} d\gamma \right) d\xi = \text{const} \end{gathered} \tag{2.7}$$

For a dilatant fluid (n > 1), the solution obtained in this way is invalid close to the boundary of the of the perturbed region of flow, since $|f_0'| \ll |f_1'|$ in its neighborhood.

3. We use the PLG method [8] to obtain uniformly valid approximations in the case of a dilatant fluid for small N. We represent f and η in the form

$$f = f_0(s) + Nf_1(s) + \ldots, \qquad \eta = s + N\eta_1(s) + \ldots \tag{3.1}$$

Inserting Eq. (3.1) in (1.4) and collecting terms which do not contain N we obtain an equation for $f_0(s)$ which coincides with (2.2). Thus, the solution for $f_0(s)$ coincides with (2.3) for n > 1. Collecting the terms containing N, we have

$$|f_0'|^{n-1} f_1'' + \frac{2-n}{1+n} s f_1' - |f_0'|^{n-1} f_0' \eta_1'' + \frac{n}{1+n} s f_0' \eta_1' + \frac{1}{1+n} f_0' \eta_1 = f_0 \tag{3.2}$$

We now make use of the principle that higher-order approximations do not have a higher-order singularity than the first approximation [8]. We can thus separate Eq. (3.2) into two equations for determining η_1 and f_1, respectively,

$$|f_0'|^{n-1}f_0'\eta_1'' - \frac{ns}{1+n} f_0'\eta_1' = -f_0 \tag{3.3}$$

$$|f_0'|^{n-1}f_1'' + \frac{2-n}{1+n} sf_1' = -\frac{1}{1+n}\eta_1 f_0' \tag{3.4}$$

Integrating (3.3) and taking the principle formulated above into account, we obtain

$$\eta_1 = C_3 + g(s), \qquad g(s) = \int_0^s \varphi^{-\frac{n}{n-1}} \Big(\int_{s_*}^s f_0 ds\Big) ds \tag{3.5}$$

The constant C_3 is determined below. Integrating Eq. (3.4) with the boundary conditions $f_1'(s_*) = f_1(s_*) = 0$, we have

$$f_1 = \frac{1}{n+1}\int_{s_*}^s \varphi^{\frac{2-n}{n-1}} \Big(\int_{s_*}^s \eta_1 ds\Big) ds \tag{3.6}$$

It follows from the condition $f = 1$ for $\eta = 0$ that

$$C_3 = \frac{f_1(0)}{f_0'(0)} = \Big[(n+1)C_1^{\frac{1}{n-1}} + \int_0^{s_*} \varphi^{\frac{2-n}{n-1}}(s_* - s)\, ds\Big]^{-1} \int_s^{s_*} \varphi^{\frac{2-n}{n-1}} \Big(\int_{s_*}^s g(s)\, ds\Big) ds \tag{3.7}$$

In the solution obtained, the quantities f_0' and f_1' are of the same order for $0 \le s \le s_*$. The boundary of the perturbed region as given by the first approximation is determined by the relation

$$\eta = \eta_* = \eta(s_*) = s_* + N\,[C_3 + g(s_*)] \tag{3.8}$$

For the particular case in which n = 2, the integrals in Eqs. (2.3), (3.5)-(3.8) can be evaluated in terms of elementary functions, and so we have

$$\begin{gathered} f_0 = {}^1/_{18}(s_* - s)^2(2s_* + s), \qquad s_* = 3^{2/3} \\ f_1 = {}^1/_3\{16ss_*^2 \ln 2s_* - 12s_*^2(s + s_*)\ln[{}^1/_2(s + s_*)/s_*] \\ - 4s_*(s_* + s)^2 \ln(s + s_*) + (s - s_*)^2[s_*({}^{101}/_{12} + 4\ln s_* + 3\ln 2) + {}^1/_6 s] + (s - s_*)4s_*(2s_* + s)\} \\ \eta_1 = s_*(6\ln 2 - {}^{23}/_6) - 6s_*s/(s + s_*) + 4s_*\ln(1 + s/s_*) - {}^1/_2 s \\ \eta_* = s_*[1 - N({}^{22}/_3 - 10\ln 2)] = 2.080 - 0.836N \end{gathered} \tag{3.9}$$

It is clear from (3.9) that as N increases, the quantity η_*, which specifies the law of motion of the perturbed fluid boundary, increases.

4. We shall now consider the solution of the problem for the case in which the values of N are large. Making the following change of variables in Eq. (1.4):

$$f(\eta) = F(z), \qquad z = \eta N^{1/(1+n)} \tag{4.1}$$

we obtain

$$|F'|^{n-1}F'' - F = -N^{-1}\frac{z}{1+n}F', \quad F(0) = 1, \quad F(\infty) = 0 \tag{4.2}$$

We shall look for a solution of F(z) in the form

$$F = F_0 + N^{-1}F_1 + N^{-2}F_2 + \ldots \tag{4.3}$$

Inserting Eq. (4.3) in (4.2) and collecting terms with like powers of N, we can construct equations for determining the F_i. Solving these equations with the appropriate boundary conditions, we have

$$\begin{gathered} F_0 = \zeta^{\frac{n+1}{n-1}}, \qquad \zeta = 1 + \frac{1-n}{2}\Big(\frac{2}{n+1}\Big)^{\frac{n}{n+1}} z \\ F_1 = \frac{1}{(n-1)^2}\big(\zeta^{\frac{2}{n-1}} - \zeta^{\frac{1+n}{n-1}}\big) + \frac{2(1+n)}{(1-n)^2(3+n)}\zeta^{\frac{2}{n-1}}\ln\zeta \end{gathered} \tag{4.4}$$

In the case of a pseudoplastic fluid $0 \leq z \leq \infty$, while for a dilatant fluid

$$0 \leqslant z \leqslant z_* = \frac{2}{n-1}\left(\frac{n+1}{2}\right)^{n/(1+n)}$$

However, if Eqs. (4.4) give uniformly accurate approximations for a pseudoplastic fluid, the approximations obtained for a dilatant fluid are valid only outside the neighborhood $z = z_*$.

Inside the region $z = z_*$, i.e., in the neighborhood of the perturbed fluid boundary, $|F_0'| \ll |F_1'|$ and so the expansion (4.2) for a dilatant fluid is invalid in this region.

5. We apply the method of internal and external expansions [8] to obtain uniformly exact approximations within the whole region of flow for a dilatant fluid. Instead of the function F(z), we now introduce the inverse function z(F), which satisfies the following equation and boundary conditions:

$$z''|z'|^{-2-n} - F = -N^{-1}z\,[(1+n)\,z']^{-1}$$
$$z(1) = 0, \qquad z'(0) = -\infty \tag{5.1}$$

We look for the external expansion of the function z(F) in the form

$$z = z_0 + N^{-1}z_1 + N^{-2}z_2 + \ldots \tag{5.2}$$

Inserting Eq. (5.2) in (5.1) and collecting terms with like powers of N, we obtain equations for the z_i, which can be solved to give

$$z_0 = \left(\frac{1+n}{2}\right)^{-1/(1+n)}\frac{1+n}{n-1}\,[1 - F^{(n-1)/(n+1)}]$$
$$z_1 = \left(\frac{1+n}{2}\right)^{-1/(1+n)}\frac{1}{n-1}\left[\frac{2}{n+3}\ln F - \frac{1}{n-1}\,(F^{(n-1)/(n+1)} - 1)\right] \tag{5.3}$$

It is clear from (5.3) that ($n > 1$) for $F \to 0, z_1 \to -\infty$ for dilatant fluids while z_0 remains finite. We must thus construct the internal expansion for z(F) in the region $F_0 = 0$, which matches with the external expansion, according to the principle enunciated in [8]. We now introduce the internal variables as follows:

$$z = Z_* + ZN^{-\alpha}, \qquad F = \Phi N^{-\beta}$$
$$(Z_*,\ \alpha,\ \beta = \mathrm{const}) \tag{5.4}$$

The indices α and β are chosen so that in the internal region in Eq. (5.1) the term on the right is of the same order as the terms on the left, i.e., the inertial term in Eq. (5.1) becomes the dominant term in the internal region. We then have for α and β

$$\alpha = 1, \qquad \beta = (1+n)\,/\,(1-n) \tag{5.5}$$

Rewriting Eq. (5.1) in terms of the internal variables, we have

$$|Z'|^{-2-n}\,Z'' - \Phi = -(Z_* + ZN^{-1})\,[(1+n)\,Z']^{-1} \tag{5.6}$$

The boundary condition for the function $Z(\Phi)$ is

$$Z'(0) = -\infty \tag{5.7}$$

and there is also the condition for matching of internal and external expansions.

Writing the external expansion (5.2) and (5.3) in terms of the internal variables (5.4), we obtain

$$z = \left(\frac{1+n}{2}\right)^{-\frac{1}{1+n}}\frac{1+n}{n-1}\left[1 - \frac{2}{(n-1)(n+2)}\frac{\ln N}{N}\right] + \left(\frac{1+n}{2}\right)^{-\frac{1}{1+n}}$$
$$\times\left[-\frac{1+n}{n-1}\,\Phi^{\frac{n-1}{n+1}}\frac{2\ln\Phi}{(n+3)(n-1)} + \frac{1}{(n-1)^2}\right]N^{-1} + o(N^{-1}) \tag{5.8}$$

If $Z(\Phi)$ is represented in the form

$$Z = Z_0 + o(1) \tag{5.9}$$

Eqs. (5.4), (5.8), and (5.9) give

$$Z^* = \left(\frac{1+n}{2}\right)^{-\frac{1}{1+n}} \frac{n+1}{n-1}\left[1 - \frac{2\ln N}{(n-1)(n+3)N}\right] \tag{5.10}$$

$$Z_0 \to \left(\frac{1+n}{2}\right)^{-\frac{1}{1+n}}\left[-\frac{n+1}{n-1}\Phi^{\frac{n-1}{n+1}} + \frac{2\ln\Phi}{(n+3)(n-1)} + \frac{1}{(n-1)^2}\right] \text{ for } \Phi\to\infty \tag{5.11}$$

Equations (5.10) and (5.11) show that not only terms containing powers of N should appear in the expansion of the quantity z, but also terms containing ln N. An equation for Z_0 is obtained from Eq. (5.6)

$$|Z_0'|^{-2-n}Z_0'' - \Phi = [{}^1/_2(1+n)]^{-1/(n+1)}[(1+n)Z_0']^{-1} \tag{5.12}$$

If Z_0' is taken as the required function, then (5.12) becomes a first-order equation and can be integrated numerically with the initial condition (5.7). Moreover, we can determine $Z_0(\Phi)$ using condition (5.11).

Equation (5.11) gives the first terms in the expansion of $Z_0(\Phi)$ for large values of the argument. The remaining terms in the expansion can easily be obtained from Eq. (5.12). The function $Z_0(\Phi)$ has the following expansion in the neighborhood of $\Phi = 0$:

$$Z_0(\Phi) = Z_0(0) + A_1\Phi^m + A_2\Phi^{2m} + A_3\Phi^{3m} + \ldots \tag{5.13}$$

where $m = (n-1)/n$, and the coefficients A_i are determined uniquely from (5.12) and (5.7).

6. The quantity η_*, which determines the boundary of the perturbed region, can be represented in the following form from the first two approximations

$$\eta_* = N^{-\frac{1}{1+n}}\left\{\left(\frac{1+n}{2}\right)^{-\frac{1}{1+n}}\frac{n+1}{n-1}\left[1 - \frac{2\ln N}{(n-1)(n+3)N}\right] + \frac{Z_0(0)}{N}\right\} \tag{6.1}$$

Thus, we see that, as N increases, the quantity η_* decreases. Calculations carried out for the case $n = 2$ give $Z_0(0) = -0.073$. For large values of N, the frictional stress on the plate τ_w is determined from the equation

$$\tau_w = \left(\frac{n+1}{2n}H^2\mu^2\sigma U^2 k^{1/n}\right)^{n/(n+1)}\left[1 + N^{-1}\frac{1}{(n+1)(3+n)}\right]^n \tag{6.2}$$

Thus, with the passage of time τ_w tends to zero as $t^{-n/(n+1)}$. We note that Eq. (6.2) is valid for power-law non-Newtonian fluids (n = 1).

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